

A Floating Random-Walk Solution for the Transverse Magnetic Electromagnetic Problem: A Homogeneous Benchmark

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Abstract

In this paper, we present the floating random-walk solution of a transverse magnetic electromagnetic problem. In a previous work, we have developed a floating random-walk algorithm suited to the solution of Maxwell-Helmholtz's equations in heterogeneous problem domains. The task of developing an approximate expression for the finite-domain Green's function for Helmholtz's equations in heterogeneous problem domains had been accomplished with the help of a novel use of iterative perturbation theory. We had applied the algorithm to the skin-effect and transverse electric electromagnetic problems in both homogeneous and heterogeneous problem domains. In this work, we extend this algorithm to a transverse magnetic electromagnetic problem and present the preliminary results for a homogeneous benchmark. We believe that with additional development this algorithm will be useful for the electromagnetic analysis of IC interconnects at high frequencies.

Index Terms- Floating random-walk algorithm, IC interconnects, Green's function, Iterative perturbation.

1. Introduction

Recent advances in digital IC interconnect technology have led to multi-GHz operating frequencies. At such high frequencies, electromagnetic effects such as skin-effect loss, frequency-dependent inductance and capacitance, slow-wave substrate coupling, distributed transmission-line propagation, and high-frequency radiation play a very important role in the electrical performance of circuits. Fortunately, electromagnetic effects in nature are described by Maxwell's equations. As a result, the semiconductor device industry has placed a considerable emphasis in developing numerical solvers for Maxwell's equations.

In a previous work [1], we have developed a floating random-walk [2] algorithm for the solution of Helmholtz equation in heterogeneous problem domains. This algorithm requires no discretization meshing of either the volume or the surface of the problem domains and hence memory requirements are expected to be orders of magnitude less than discretization-based methods such as the finite-element [3] or the method of moments [4]. The algorithm is completely parallelizable and a linear increase in computational speed is achieved with the increase in the number of processors. But, before we describe the details of this floating random-

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walk algorithm, we will first describe the transverse magnetic (TM) problem that we are going to solve in this work.

2. The transverse magnetic problem

For the TM problem [5], the electric field and the direction of propagation are in the xy plane, while the magnetic field is in the z direction. The electric field components have possible spatial variations in the xy plane; therefore, $\mathbf{E} = E_x(x, y)\hat{\mathbf{e}}_x + E_y(x, y)\hat{\mathbf{e}}_y$ where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are unit vectors in the x and y direction. The resulting field equation can be written as

$$\nabla^2 \mathbf{E} + \gamma^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mathbf{0}, \quad (1)$$

where the propagation constant, γ is given by the relation

$$\gamma^2 = \mu\epsilon\omega^2 - j\omega\sigma\mu. \quad (2)$$

Equation (1) is not of the vector-Helmholtz type; the components of the electric field are coupled because of the $\nabla(\nabla \cdot \mathbf{E})$ term. To achieve uncoupling of these components, we transform the problem to a vector potential formulation. The four-vector potential (ϕ, \mathbf{A}) is a complete description of the electromagnetic field, where ϕ is the scalar potential and \mathbf{A} is the vector potential. In the divergence-free Coulomb gauge (i.e., $\nabla \cdot \mathbf{A} = 0$), each component of \mathbf{A} independently satisfies its own scalar Helmholtz equation

$$\nabla^2 \mathbf{A} + \gamma^2 \mathbf{A} = \mu \hat{\mathbf{J}}. \quad (3)$$

Above, $\hat{\mathbf{J}} = \hat{\sigma} \nabla \phi$ and $\hat{\sigma} = \sigma(x, y) + j\omega\epsilon(x, y)$, where $\hat{\mathbf{J}}, \omega, \sigma, \epsilon$ and μ are the current density, frequency, conductivity, permittivity and permeability respectively. As a result, we need to solve a scalar Helmholtz equation for each component of \mathbf{A} . Harmonic time dependence with a complex propagation constant has been assumed in deriving equation (1) to equation (3). But before we describe the specifics of our benchmark problem, we will briefly describe our previously developed iterative perturbation-based floating random-walk algorithm.

3. The floating random-walk algorithm

The objective of the floating random-walk algorithm is to solve for equation (3), by solving the scalar Helmholtz equation for each of the components of the vector-potential. The Green's function equation for equation (3) is given by

$$\nabla_r^2 G(\mathbf{r} | \mathbf{r}_o) + \gamma^2(\mathbf{r}) G(\mathbf{r} | \mathbf{r}_o) = \delta(\mathbf{r} - \mathbf{r}_o), \quad (4)$$

where $G(\mathbf{r} | \mathbf{r}_o)$ is the 2D Green's function at position coordinate \mathbf{r} with a Dirac delta function applied at \mathbf{r}_o . If we impose Dirichlet boundary condition on $G(\mathbf{r} | \mathbf{r}_o)$, we have $G(\mathbf{r} | \mathbf{r}_o) = 0$, for $\mathbf{r} \in S$, where S is a circle of radius R . If we normalize R to unity in equation (4), the Green's function equation reduces to

$$\nabla_{\mathbf{r}}^2 G(\mathbf{r} | \mathbf{r}_o) + R^2 \gamma^2(\mathbf{r}) G(\mathbf{r} | \mathbf{r}_o) = \delta(\mathbf{r} - \mathbf{r}_o). \quad (5)$$

If we use iterative perturbation theory, the zeroth order approximation $G^{(0)}(\mathbf{r} | \mathbf{r}_o)$ is obtained by substituting $\gamma = 0$, and is governed by the differential equation

$$\nabla_{\mathbf{r}}^2 G^{(0)}(\mathbf{r} | \mathbf{r}_o) \approx \delta(\mathbf{r} - \mathbf{r}_o). \quad (6)$$

By iteration, we get a differential equation governing the first-order approximation $G^{(1)}(\mathbf{r} | \mathbf{r}_o)$, which is expressed as

$$\nabla_{\mathbf{r}}^2 G^{(1)}(\mathbf{r} | \mathbf{r}_o) \approx \delta(\mathbf{r} - \mathbf{r}_o) - R^2 \gamma^2(\mathbf{r}) G^{(0)}(\mathbf{r} | \mathbf{r}_o). \quad (7)$$

The solution to equation (6) is the well known Green's function to Laplace's equation in polar coordinates and is given by [6]

$$G^{(0)}(\mathbf{r} | \mathbf{r}_o) = \frac{1}{4\pi} \ln \left[\frac{\rho^2 + \rho_o^2 - 2\rho\rho_o \cos(\theta - \theta_o)}{1 + \rho^2 \rho_o^2 - 2\rho\rho_o \cos(\theta - \theta_o)} \right]. \quad (8)$$

Thus, $G^{(1)}(\mathbf{r} | \mathbf{r}_o)$ can be written as

$$G^{(1)}(\mathbf{r} | \mathbf{r}_o) = \oint_S dS' G^{(0)}(\mathbf{r}' | \mathbf{r}_o) [\delta(\mathbf{r}' - \mathbf{r}_o) - R^2 \gamma^2(\mathbf{r}') G^{(0)}(\mathbf{r}' | \mathbf{r}_o)]. \quad (9)$$

Equation (9) can be simplified to

$$G^{(1)}(\mathbf{r} | \mathbf{r}_o) = G^{(0)}(\mathbf{r} | \mathbf{r}_o) - \oint_S dS' R^2 \gamma^2(\mathbf{r}') G^{(0)}(\mathbf{r}' | \mathbf{r}) G^{(0)}(\mathbf{r}' | \mathbf{r}_o). \quad (10)$$

We assume that $\gamma^2(\mathbf{r})$ is piecewise constant in θ , with possible discontinuities at $\theta = 0, \pi/2, \pi$ and $3\pi/2$. We also set $\mathbf{r}_o = 0$ ($\rho_o = 0, \theta_o = 0$). As a result, $G^{(1)}(\mathbf{r})$ becomes [1]

$$G^{(1)}(\mathbf{r}) = \frac{1}{2\pi} \ln \rho - \frac{R^2}{8\pi^2} \int_0^{2\pi} d\theta' \int_0^1 d\rho' \sum_q \gamma_q^2 \ln \left[\frac{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')}{1 + \rho^2 \rho'^2 - 2\rho\rho' \cos(\theta - \theta')} \right] \rho' \ln \rho'. \quad (11)$$

Above, $\gamma_q = \gamma(\mathbf{r})$ for \mathbf{r} in the q th quadrant. Interchanging the order of summation and integration operations, equation (11) leads to

$$G^{(1)}(\mathbf{r}) = \frac{1}{2\pi} \ln \rho - \frac{R^2}{8\pi^2} \sum_q \int_0^1 d\rho' \int_{(q-1)\pi/2}^{q\pi/2} d\theta' \gamma_q^2 \ln \rho' \ln \left[\frac{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')}{1 + \rho^2 \rho'^2 - 2\rho\rho' \cos(\theta - \theta')} \right]. \quad (12)$$

$G^{(1)}(\mathbf{r})$ is an approximate expression for the volume Green's function for Helmholtz equation, including terms up to the first order. Using Green's theorem [6] and imposing Dirichlet boundary conditions, one can show that a particular component, say A_x of the vector-potential given by equation (3) at any point \mathbf{r}_0 is

$$A_x(\mathbf{r}_0) = \oint_C [\hat{\mathbf{n}} dc \cdot \nabla_{\mathbf{r}} G^{(1)}(\mathbf{r} | \mathbf{r}_0)] A_x(\mathbf{r}) + \oint_S dS G^{(1)}(\mathbf{r} | \mathbf{r}_0) \mu J. \quad (13)$$

The first integral in equation (13) is over the circumference of a unit circle centered at \mathbf{r}_0 . The second integral in equation (13) is over the surface of the same unit circle. Now, $\hat{\mathbf{n}} dc \cdot \nabla_{\mathbf{r}} \rightarrow \rho d\theta \partial / \partial \rho$ at $\rho=1$ in polar coordinates. Taking the derivative of $G^{(1)}(\mathbf{r})$ yields

$$\left(\frac{\partial G^{(1)}}{\partial \rho} \right)_{\rho=1} = \frac{1}{2\pi} - \frac{R^2}{4\pi^2} \sum_q \int_0^1 d\rho' \int_{(q-1)\pi/2}^{q\pi/2} d\theta' \frac{\gamma_q^2 \ln \rho' (1 - \rho'^2) \rho'}{\rho'^2 - 2\rho' \cos(\theta - \theta') + 1}. \quad (14)$$

The vector quantity $\nabla_{\mathbf{r}} G^{(1)}(\mathbf{r} | \mathbf{r}_0)$ is the "surface Green's function", and equation (13) together with equation (14) form the basis of a floating random-walk algorithm for solving a scalar Helmholtz equation subject to Dirichlet boundary conditions. The random-walks start at a point, where we need to estimate a particular component of the vector-potential given by equation (3). The random-walks propagate as "hops" of different sizes from circle centers to circumferences, consistent with a statistical interpretation [2] of (13). Maximally sized circles, subject to limitations imposed by iterative perturbation theory, are used with hop-location probability rules again consistent with (13). We define, with each hop, a numerical weight factor derived from (12). The product of these weight factors over a walk, multiplied by the solution at the problem boundary—where the walk must terminate—gives a statistical estimate for the component of the vector-potential at the random-walk starting point. All this, again, is entirely consistent with a statistical interpretation of (13). We can thus obtain an accurate statistical estimate for the component of the vector-potential by averaging over a statistically large number of random-walks. Mathematically, we can write such an estimate

$$\bar{A} \approx \frac{1}{N} \sum_{n=1}^N A^{(n)}, \quad (15)$$

where N is the number of walks and $A^{(n)}$ is the n th-walk estimate. At this point, it is easy to see the relative advantages of this algorithm. There was no need of any discretization of the problem domain leading to lower memory requirements in comparison with discretization based algorithms. It can also be noted that the random-walks are independent of each other, which makes the algorithm completely parallelizable. In the next section, we will describe the benchmark problems that have been solved in this work.

4. Homogeneous Benchmark Problems

Two materially homogeneous problems have been chosen in an insulating and conducting material domain respectively for a problem geometry shown in figure 1. In each of these problems, the values of the x -component of the vector-potential, A_x at gray-region boundaries are zero at top, bottom, left and right edges. We have chosen a forcing function that varies sinusoidally in the x and y direction. The motivation behind such a forcing function is that any arbitrary forcing term (that is piece-wise continuous) can be decomposed into an infinite number of sinusoids with the help of a Fourier expansion. The analytical and random-walk results are plotted for solution points along the centerline starting with $x = 0$ at the left edge in figure 2 and figure 3 and they are seen to be in good agreement. Table 1 summarizes our computational results. From Table 1, it can be noted that our predicted statistical error, is consistent with the mean absolute error.

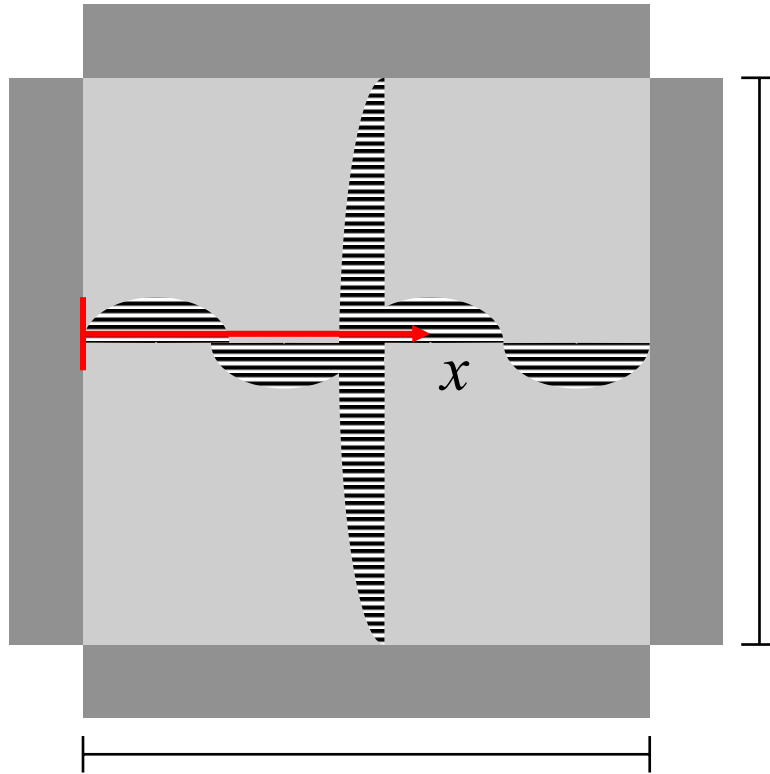


Figure 1: 2D, TM homogeneous test problem, the geometry. The computational region is represented in white, and the boundary regions where vector potential is known, are represented in gray. $A_x = 0$ in the top, bottom, left and right boundary regions. A sinusoidal forcing term is applied in x and y directions: $f(x, y) = \sin(4\pi x / L) \sin[\pi y / (2L)]$.

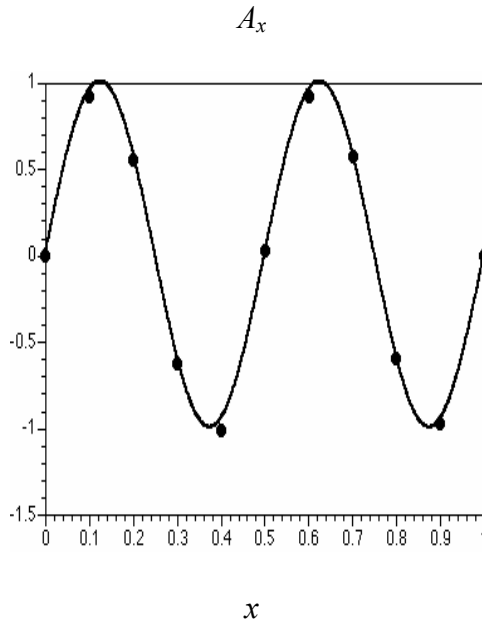


Figure 2: 2D, TM homogeneous problem in insulating medium. Square domain with $L = 100 \mu\text{m}$, $\epsilon_r = 2.7$. The solid line represents the exact analytical solution at $\omega = 100 \text{ GHz}$. The dots represent random-walk solution points at $\omega = 100 \text{ GHz}$. A normalized length scale has been used.

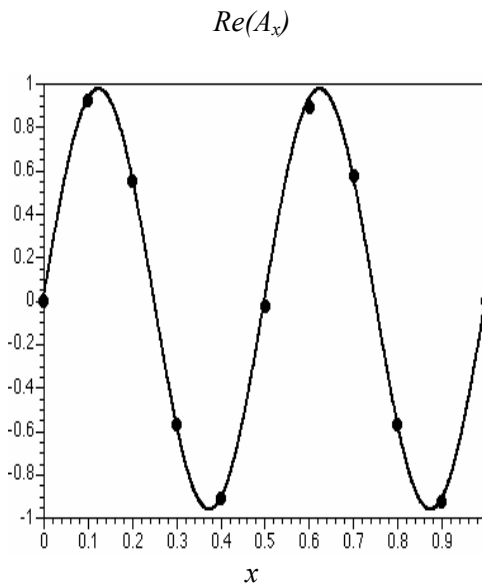


Figure 3a: 2D, TM homogeneous problem in conducting domain, the real part of the solution. Square domain with $L = 10 \mu\text{m}$, $\rho = 1.8 \mu\Omega\text{-cm}$. The solid line represents the exact analytical solution at $\omega = 1 \text{ GHz}$. The dots represent the random-walk solution points at $\omega = 1 \text{ GHz}$. A normalized length scale has been used.

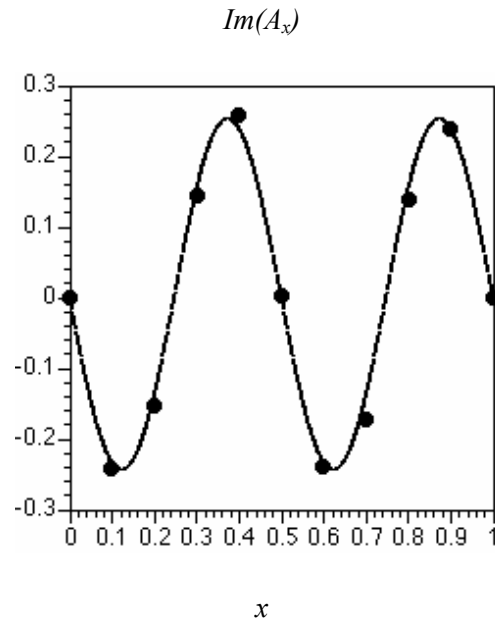


Figure 3b: 2D, TM homogeneous problem in conducting domain, the imaginary part of the solution. Square domain with $L = 10 \mu\text{m}$, $\rho = 1.8 \mu\Omega\text{-cm}$. The solid line represents the exact analytical solution at $\omega = 1 \text{ GHz}$. The dots represent the random-walk solution points at $\omega = 1 \text{ GHz}$. A normalized length scale has been used.

Problem #	$f \text{ (GHz)}$	Random Walks Per Point ($\times 10^3$)	Mean Statistical error	Mean Absolute Error
<i>Homogeneous Dielectric</i>	400	10	0.02	0.02
<i>Homogeneous Conductor</i>	1.0	10	$0.02+0.01i$	$0.01+0.01i$

Table 1: Numerical results for the TM homogeneous benchmark problems.

5. Conclusion and direction of future work

Summarizing, we have tested our iterative-perturbation-theory based 2D floating random-walk algorithm on a homogeneous TM problem. Excellent agreement has been obtained between our numerical results and known analytical solutions. Extension of this algorithm to a 2D heterogeneous TM problem as well as the development of a 3D version of this algorithm will form the basis of future work. We believe that with additional development, this algorithm can be useful in the electromagnetic analysis of complex IC interconnect structures.

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